Fundamentals of magnetohydrodynamics

Daniel Gómez [1,2]

Email: dgomez@df.uba.ar Webpage: http://astro.df.uba.ar

(1) Instituto de Astronomia y Fisica del Espacio, CONICET, Argentina
(2) Departamento de Fisica, Universidad de Buenos Aires, Argentina

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What do we mean by magnetohydrodynamics ?

- It is a fluid-like theoretical description for the dynamics of matter
- Baryonic matter in the Universe is mostly hydrogen.
- At temperatures above 10^4 K it becomes a hydrogen plasma, i.e. a gas made of protons and electrons
- The large scale behavior of this gas can be described through fluidistic equations (Navier-Stokes).
- This fluid is made of electrically charged particles and therefore it suffers electric and magnetic forces.
- Not only that, these charges are sources of self-consistent electric and magnetic fields. Therefore, the fluid equations will couple to Maxwell’s equations.
- At small spatial scales (and fast timescales) non-fluid or kinetic effects become non-negligible.

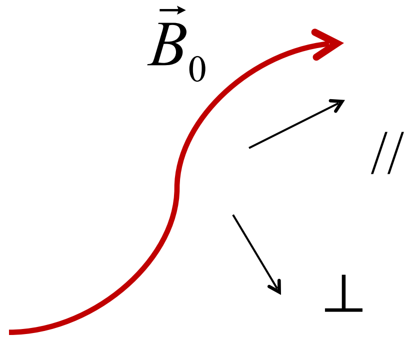
What does MHD mean?

- Many laboratory, astrophysical and space plasmas can properly be described within the theoretical framework of Magnetohydrodynamics (MHD).

- MHD is a fluidistic approach to describe the large scale dynamics of plasmas.

- It involves the Navier-Stokes equation for the motion of the plasma subjected to electric and magnetic forces, coupled to Maxwell's equations to describe the evolution of the self-consistent electric and magnetic fields.

- Physical processes that can be addressed with MHD include:
  - Magnetic reconnection
  - Magnetic confinement
  - Magnetic dynamo
  - MHD turbulence



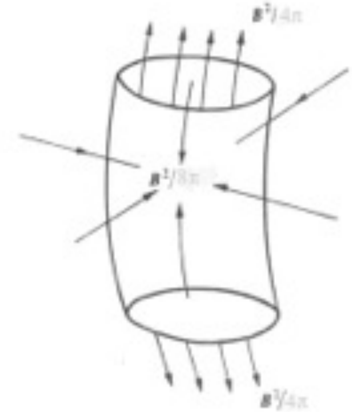
- We will also address the case of plasmas embedded in strong external magnetic fields, which allow for an approximation known as reduced MHD (RMHD).

MHD equations

- The equations for the fluid are:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \bullet (\rho \vec{u}) \qquad p = p_0 \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

$$\rho \frac{\partial \vec{u}}{\partial t} = -\rho (\vec{u} \bullet \vec{\nabla}) \vec{u} - \vec{\nabla} p + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B} + \vec{F}_{ext} + \vec{\nabla} \bullet \vec{\sigma}_{visc}$$



- We also have Ohm's law written in the fluid's reference frame

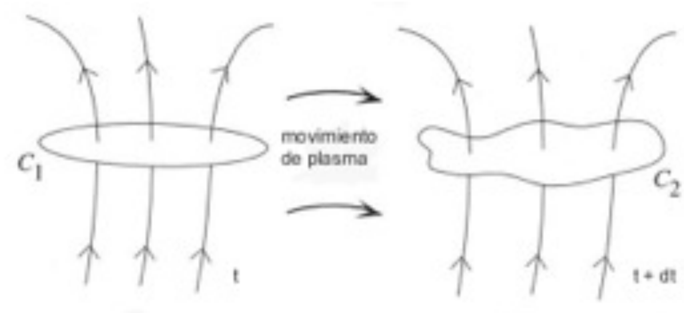
$$\frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} = \frac{1}{4\pi}(\vec{B} \cdot \vec{\nabla})\vec{B} - \vec{\nabla}\left(\frac{B^2}{8\pi}\right)$$

Magnetic pressure
and magnetic tension

$$\vec{E}^* = \vec{E} + \tfrac{1}{c} \vec{u} \times \vec{B} = \frac{1}{\sigma} \vec{J}$$

- The induction equation arises by combining Ampere's law and Ohm's law

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B}) + \eta \nabla^2 \vec{B}, \qquad \vec{\nabla} \bullet \vec{B} = 0$$



Frozen-in condition

Fluid equations for multi-species plasmas

➢ For each species s we have (Goldston & Rutherford 1995):

- Mass conservation

$$\frac{\partial n_s}{\partial t} + \vec{\nabla} \cdot (n_s \vec{U}_s) = 0$$

- Equation of motion

$$m_s n_s \frac{d\vec{U}_s}{dt} = q_s n_s (\vec{E} + \frac{1}{c}\vec{U}_s \times \vec{B}) - \vec{\nabla} p_s + \vec{\nabla} \bullet \vec{\sigma}_s + \sum_{s'} \vec{R}_{ss'}$$

- Momentum exchange rate

$$\vec{R}_{ss'} = -m_s n_s \upsilon_{ss'} (\vec{U}_s - \vec{U}_{s'})$$

➢These moving charges act as sources for electric and magnetic fields:

- Charge density (charge neutrality)

$$\rho_c = \sum_s q_s n_s \approx 0$$

- Electric current density

$$\vec{J} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = \sum_s q_s n_s \vec{U}_s$$

Two-fluid MHD equations

➢ For a fully ionized plasma with ions of mass m_i and massless electrons (since $m_e << m_i$):

- Mass conservation: $$0 = \frac{\partial n}{\partial t} + \vec{\nabla}\cdot(n\vec{U}) \quad , \qquad n_e \cong n_i \cong n$$
- Ions: $$m_i n \frac{d\vec{U}}{dt} = en(\vec{E} + \frac{1}{c}\vec{U}\times\vec{B}) - \vec{\nabla}p_i + \vec{\nabla}\bullet\vec{\sigma} + \vec{R}$$
- Electrons: $$0 = -en(\vec{E} + \frac{1}{c}\vec{U}_e\times\vec{B}) - \vec{\nabla}p_e - \vec{R}$$
- Friction force: $$\vec{R} = -m_i n \nu_{ie}(\vec{U} - \vec{U}_e)$$
- Ampere's law: $$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = en(\vec{U} - \vec{U}_e) \quad \Rightarrow \quad \vec{R} = -\frac{m\upsilon_{ie}}{e}\vec{J}$$
- Polytropic laws: $$p_i \propto n^\gamma \quad , \qquad p_e \propto n^\gamma$$
- Newtonian viscosity: $$\sigma_{ij} = \mu\,(\partial_i U_j + \partial_j U_i)$$

Hall-MHD equations

➢ The dimensionless version, for a length scale L_0, density n_0 and Alfven speed $v_A = B_0 / \sqrt{4\pi m_i n_0}$

$$\frac{d\vec{U}}{dt} = \frac{1}{\varepsilon}(\vec{E} + \vec{U} \times \vec{B}) - \frac{\beta}{n}\vec{\nabla}p_i - \frac{\eta}{\varepsilon n}\vec{J} + \nu\nabla^2\vec{U} \qquad \nu = \frac{\mu}{m_i n v_A L_0}$$

$$0 = -\frac{1}{\varepsilon}(\vec{E} + \vec{U}_e \times \vec{B}) - \frac{\beta}{n}\vec{\nabla}p_e + \frac{\eta}{\varepsilon n}\vec{J} \qquad \text{where} \qquad \vec{J} = \vec{\nabla} \times \vec{B} = \frac{n}{\varepsilon}(\vec{U} - \vec{U}_e)$$

➢ We define the Hall parameter $\varepsilon = \frac{c}{\omega_{pi} L_0}$

as well as the plasma *beta* $\beta = \frac{p_0}{m_i n_0 v_A^2}$ and the electric resistivity $\eta = \frac{c^2 \nu_{ie}}{\omega_{pi}^2 L_0 v_A}$

➢ Adding these two equations yields:

$$n\frac{d\vec{U}}{dt} = (\vec{\nabla} \times \vec{B}) \times \vec{B} - \beta\vec{\nabla}(p_i + p_e) + \nu\nabla^2\vec{U}$$

Hall-MHD equations

➢ On the other hand, using

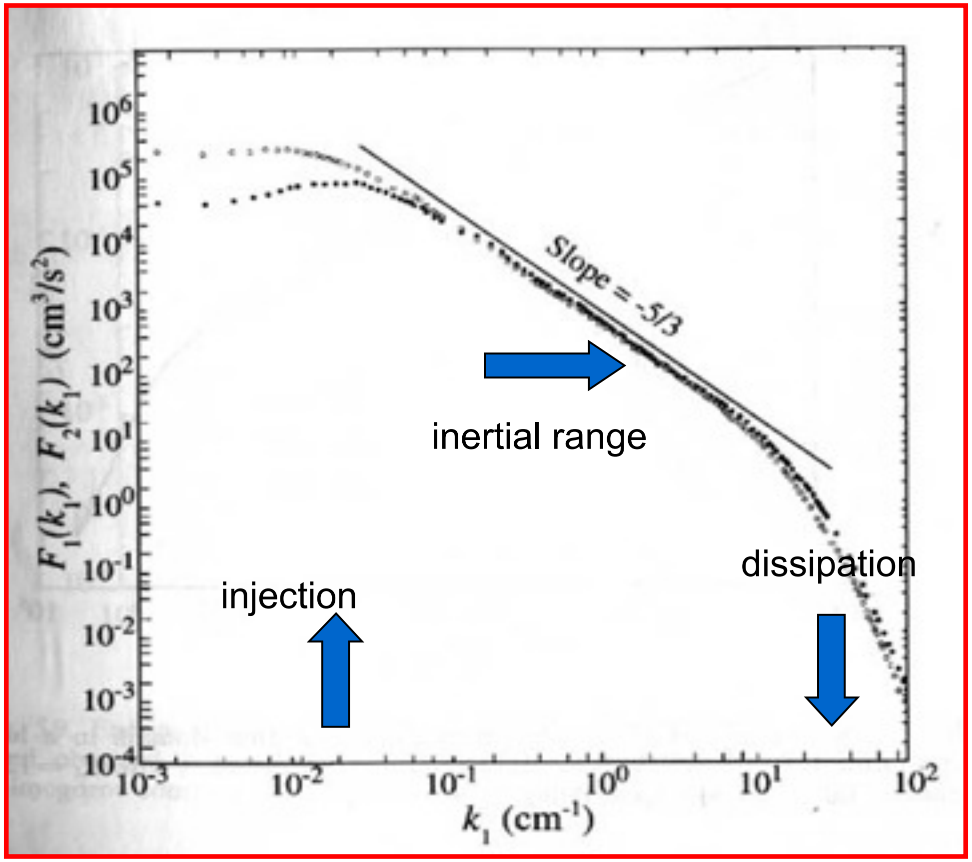
$$\left.\begin{aligned} \vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}\right\} \rightarrow \frac{\partial \vec{A}}{\partial t} = (\vec{U} - \frac{\varepsilon}{n}\vec{\nabla} \times \vec{B}) \times \vec{B} - \vec{\nabla}\phi + \frac{\varepsilon\beta}{n}\vec{\nabla}p_e - \frac{\eta}{n}\vec{\nabla} \times \vec{B}$$

MHD turbulence

- Energy cascade
  - energy flux toward high k
  - vortex breakdown

- Scale invariance
  - energy flux in k: → $\varepsilon_k \approx \frac{u_k^2}{\tau_k}$
  - energy power spectrum: → $E_k \approx \frac{u_k^2}{k}$

$$\tau_k \approx \frac{1}{k u_k}\ ,\ \varepsilon_k \approx \frac{u_k^2}{\tau_k} = const.$$

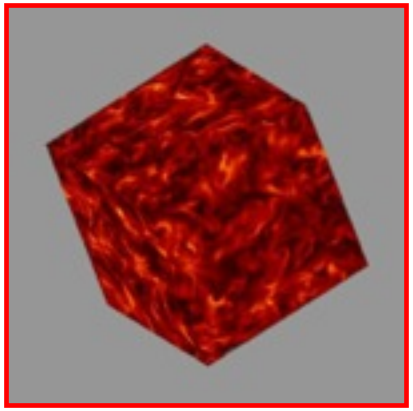


- Therefore → $E_k \approx \frac{u_k^2}{k} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$

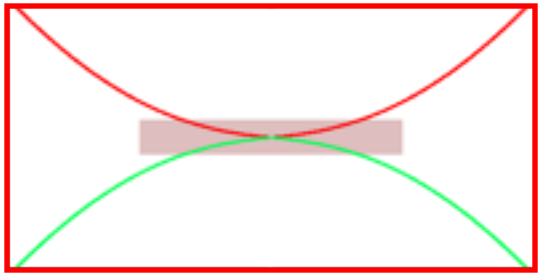
Kolmogorov spectrum (K41)

Various applications

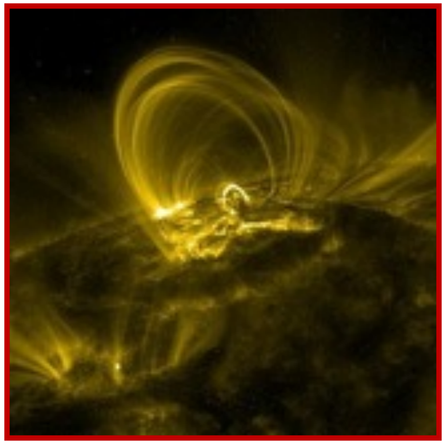
- We studied a number of astrophysical problems, within the general framework of MHD:
- 3D Hall-MHD turbulent dynamos.
(Mininni, Gomez & Mahajan 2003, 2005;
Gomez, Dmitruk & Mininni 2010)



- 2.5 D Hall-MHD magnetic reconnection
in the Earth magnetosphere
(Morales, Dasso & Gomez 2005, 2006)



- 3D HD helical fluid turbulence
(Gomez & Mininni 2004)
- RMHD heating of solar coronal loops
(Dmitruk & Gomez 1997, 1999)



- RHMHD turbulence in the solar wind
(Martin, Dmitruk & Gomez 2010, 2012)
- Hall magneto-rotational instability in accretion disks
(Bejarano, Gomez & Brandenburg 2011)

Coronal Heating

- The solar corona is a topologically complex array of loops (TRACE movie 171 A)
- Coronal loops are magnetic flux tubes with their footpoints anchored deep in the convective region.
- They confine a tenuous and hot plasma. Typical densities are $n = 10^9$ cm^{-3} and temperatures are $T = 2\text{-}3.10^6$ K.

- The magnetic field provides not just the confinement of the plasma, but also the energy to heat it up to coronal temperatures (Parker 1972, 1988; van Ballegooijen 1986; Einaudi et al. 1996).
- One of the key ingredients is the free energy available in the sub-photospheric convective region. Convective motions move the footpoints of fieldlines, thus building up magnetic stresses.
- However, the typical length scale of these magnetic stresses is way too large for the Ohmic dissipation to do the job, since

$$\tau_{diss} \approx \ell^2 / \eta$$

Reduced MHD (RMHD)

- Reduced MHD is a self-consistent approximation of the full MHD equations whenever:
  (a) one component of the magnetic field is much stronger than the others and,
  (b) spatial variations are smoother along than across (Strauss 1976).

$$\partial_t a = v_A \partial_z \varphi + [\varphi, a] + \eta \nabla_\perp^2 a$$

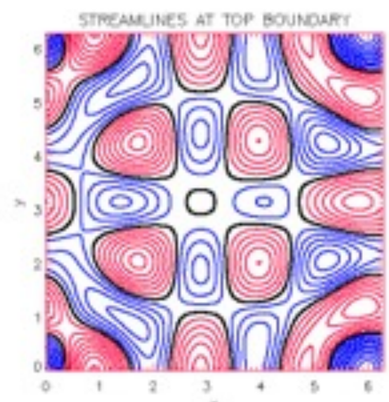
$$\partial_t \omega = v_A \partial_z j + [\varphi, \omega] - [a, j] + \eta \nabla_\perp^2 \omega$$

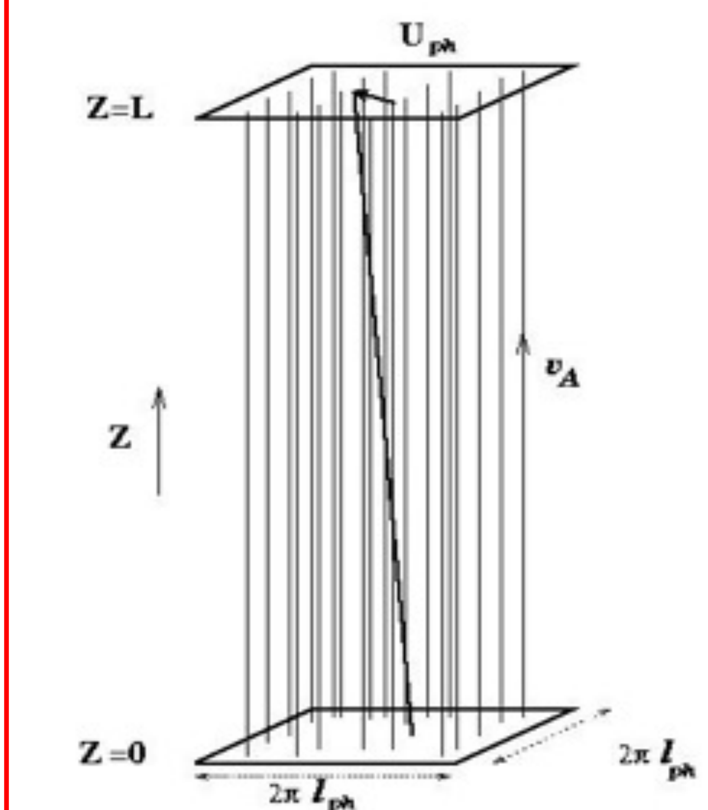
$$\vec{b} = v_A \hat{z} + \vec{\nabla}_\perp \times (a\, \hat{z}) \quad , \quad \vec{u} = \vec{\nabla}_\perp \times (\varphi\, \hat{z})$$

$$\omega = -\nabla_\perp^2 \varphi \quad , \quad j = -\nabla_\perp^2 a$$

- These equations describe the evolution of the velocity field (u) and magnetic field (b) inside the box, assuming periodic boundary conditions at the sides.

- We enforce stationary velocity fields (U_{ph}) at the top plate. Granules size is ℓ_{ph} and the turnover time is t_{ph}.





Numerical simulations

- We integrate the RMHD eqs. numerically, using a spectral scheme in the perpendicular directions and finite differences along the (much smoother) direction z (Gomez, Milano and Dmitruk 2000; also Dmitruk, Gomez & Matthaeus 2003)

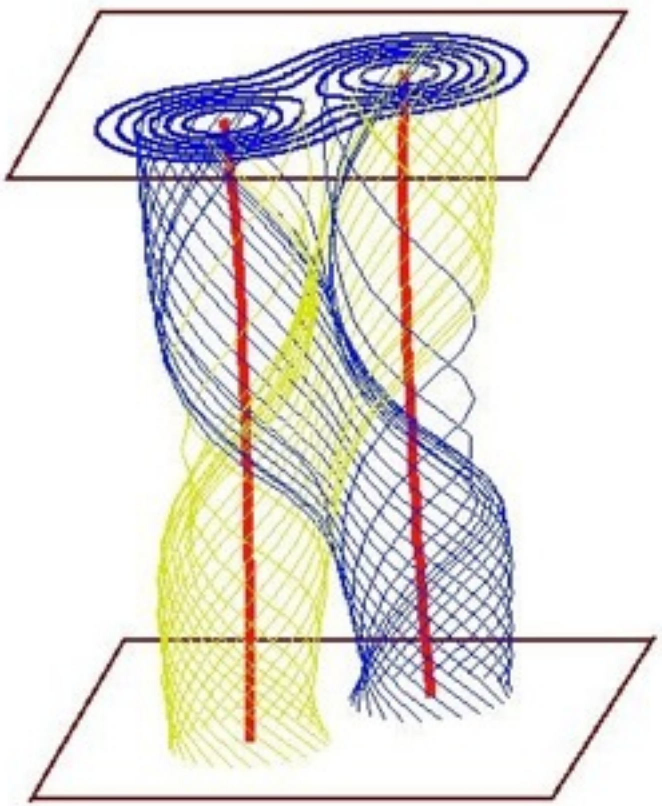
- We show results from 512x512x40 runs performed in (CAPS), our linux cluster with 80 cores

- For the horizontal spatial derivatives, we use a pseudo-spectral scheme with 2/3-dealiasing. Spectral codes are well suited for turbulence studies, since they provide exponentially fast convergence. Spatial derivatives along the loop are computed using finite differences.

- Time integration is performed with a second order Runge-Kutta scheme.The time step is chosen to satisfy the CFL condition.

caps
Cluster for Astrophysical Plasma Simulations

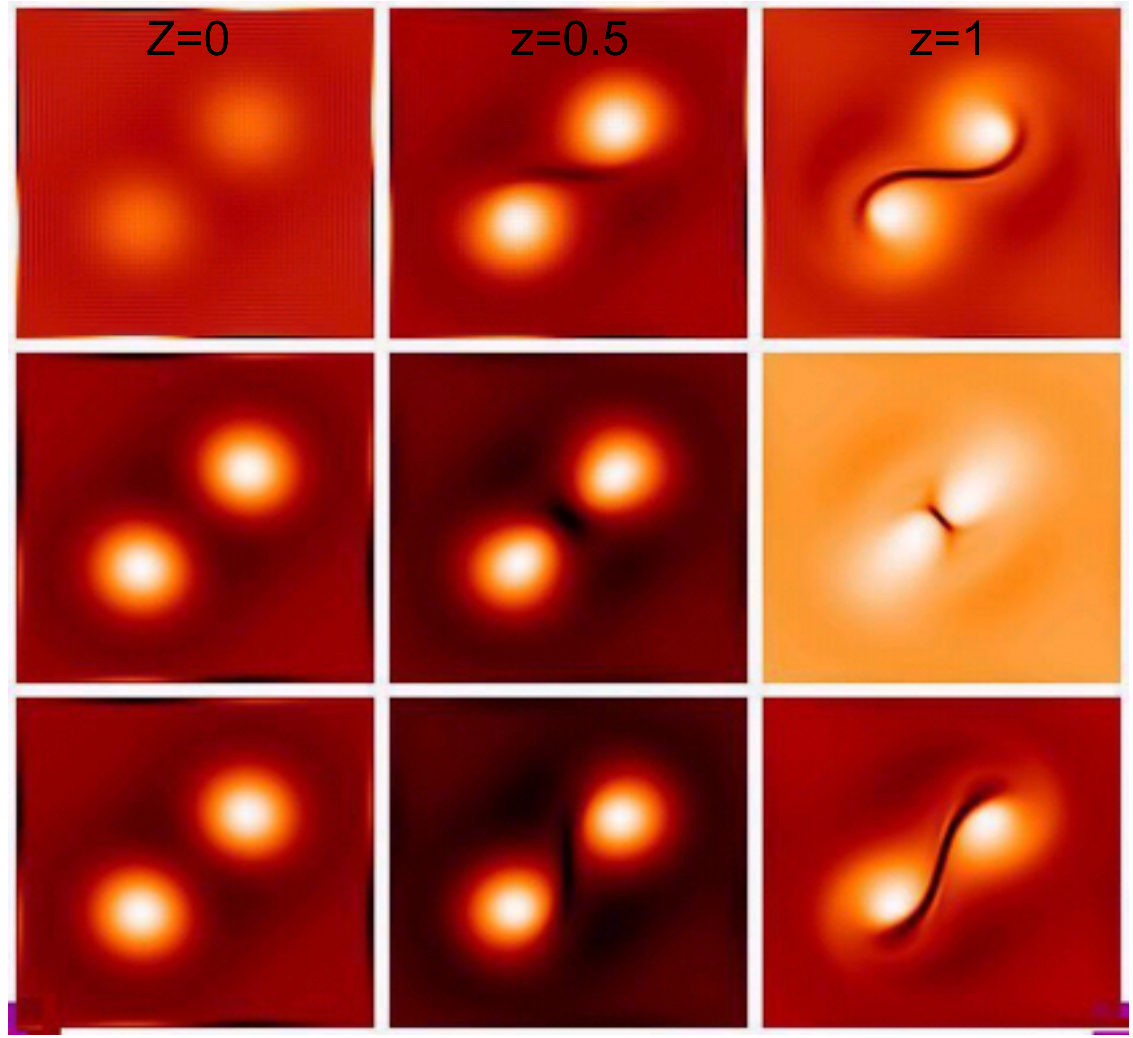
Magnetic field lines



- Fieldlines are computed for a simulation driven by two gaussian vortices at the upper boundary.
- Blue and yellow fieldlines simply correspond to different vortices.
- At early times, only the progressive twist of two flux tubes can be observed.
- As time evolves, we see how fieldlines from different vortices literally reconnect, i.e. they start in one vortex and end up on the other.

Current density distribution

Current density



time

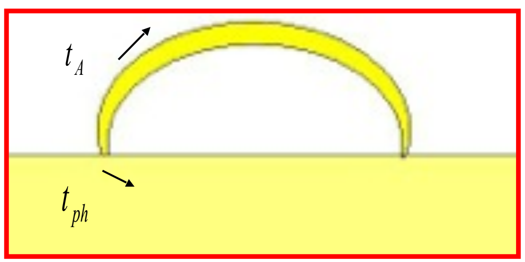
RMHD Simulations

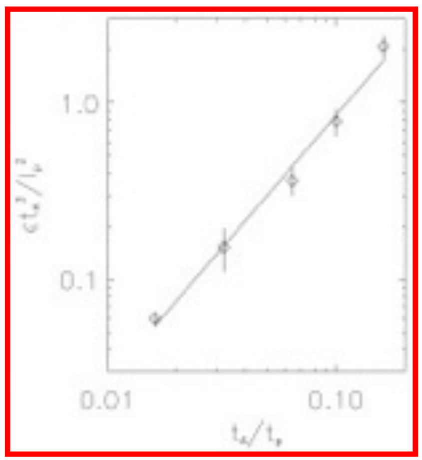
- We perform long time integrations of the RMHD equations. Lengths are in units of the photospheric convective motions (ℓ_{ph}) and times are in units of the Alfven time (t_A) along the loop.

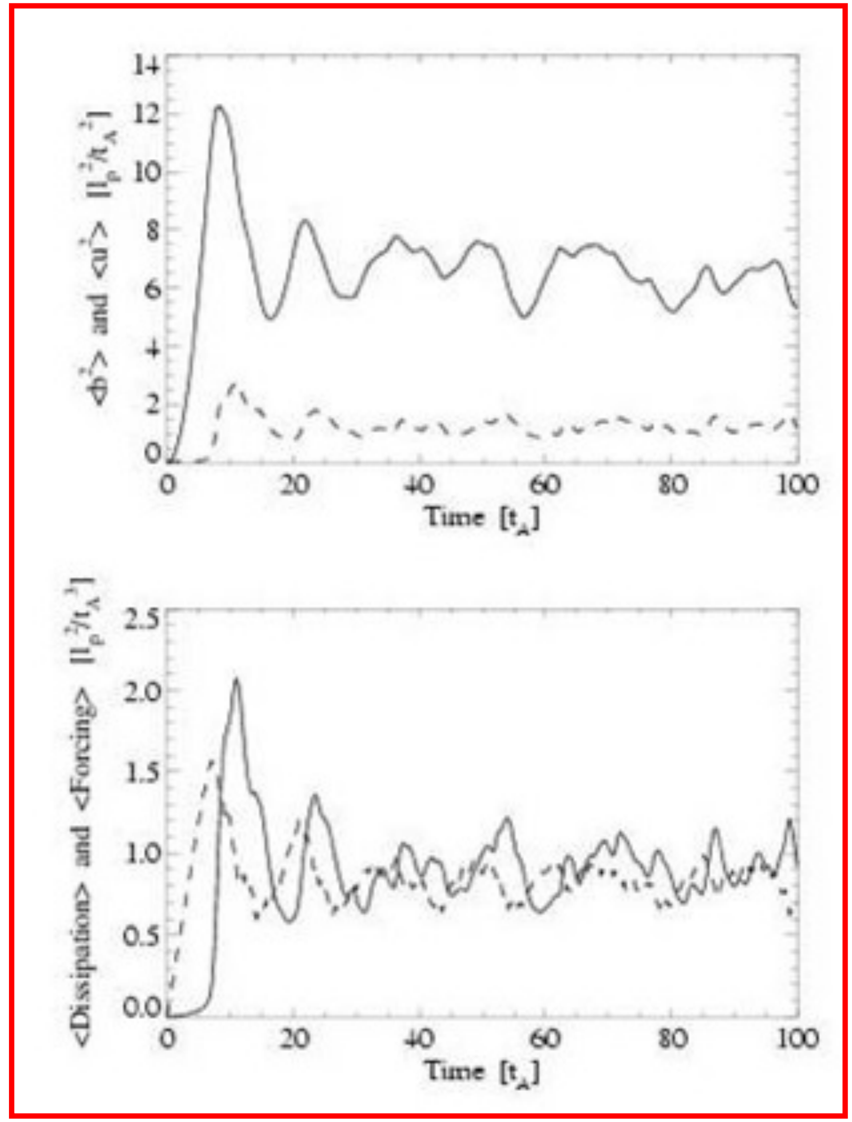
- Spatial resolution is 512x512x40 and the integration time is 4000 t_A. We use a spectral scheme in the xy-plane and finite differences along z.

- The time averaged dissipation rate is found to scale like (Dmitruk & Gómez 1999)

$$\varepsilon \approx \frac{\rho\, \ell_{ph}^2}{t_A^3} \left(\frac{t_A}{t_{ph}}\right)^{3/2}$$



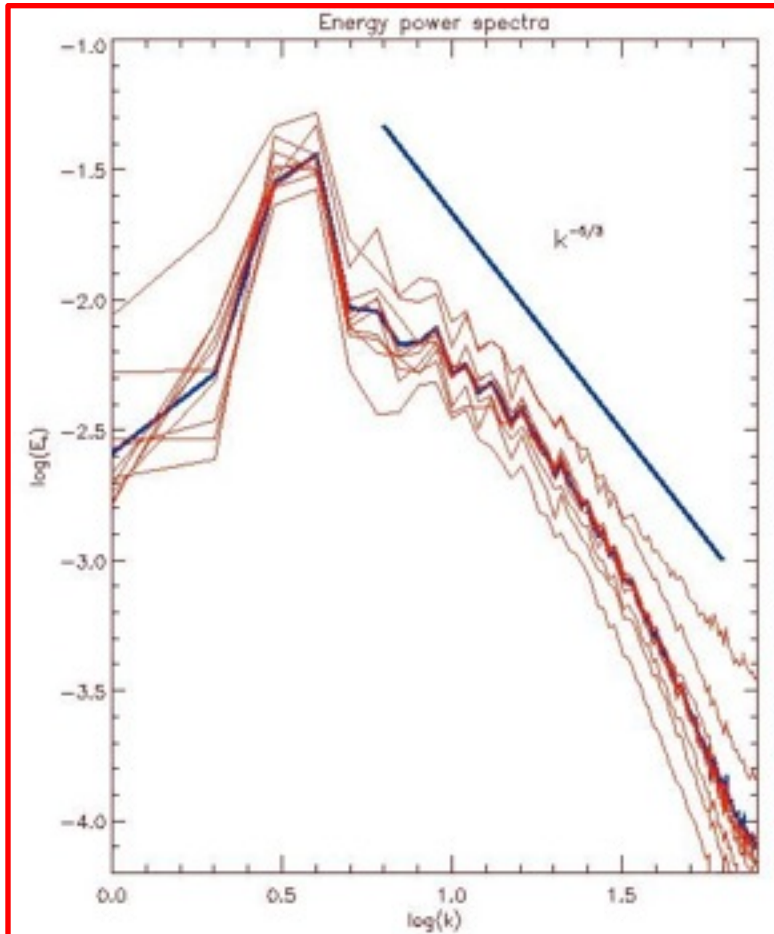




- It is essentially independent of the Reynolds number, as expected for stationary turbulence.

Energy power spectra

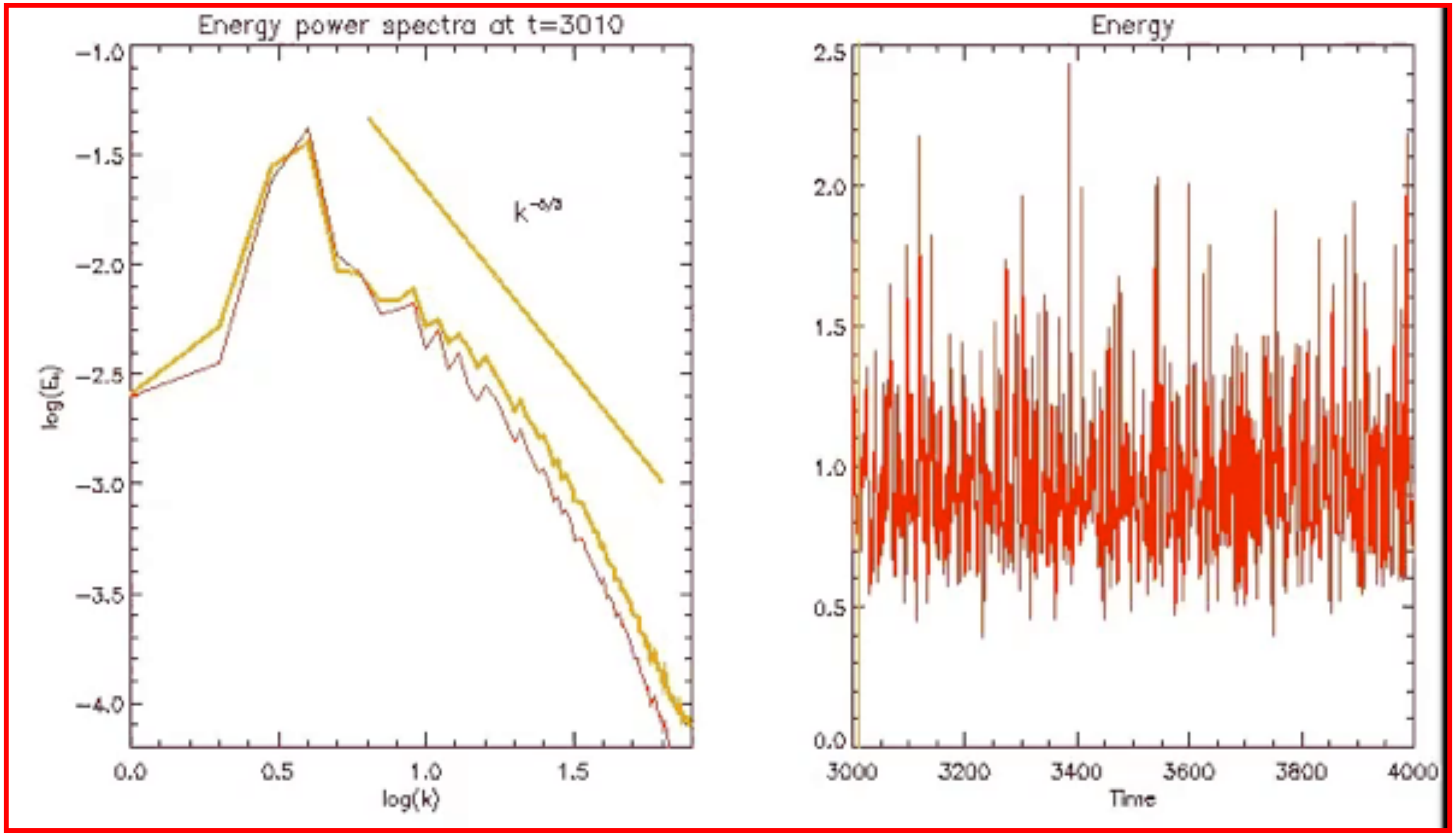
- The energy spectra are shown here. The **red lines** correspond to ten spectra taken at different times (separated by 10 t_A). The **blue trace** is the time averaged version.
- The Kolmogorov slope is displayed for reference, but the moderate spatial resolution of these runs is insufficient for a serious spectral analysis.
- Viscosity and resistivity are large enough to spatially resolve the dissipative structures properly.
- The spectra of kinetic energy (not shown) remain much smaller than magnetic energy, although they tend to equipartition at the largest wavenumbers.



$$E_k \approx \frac{u_k^2}{k} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

Kolmogorov spectrum (K41)

Energy power spectra

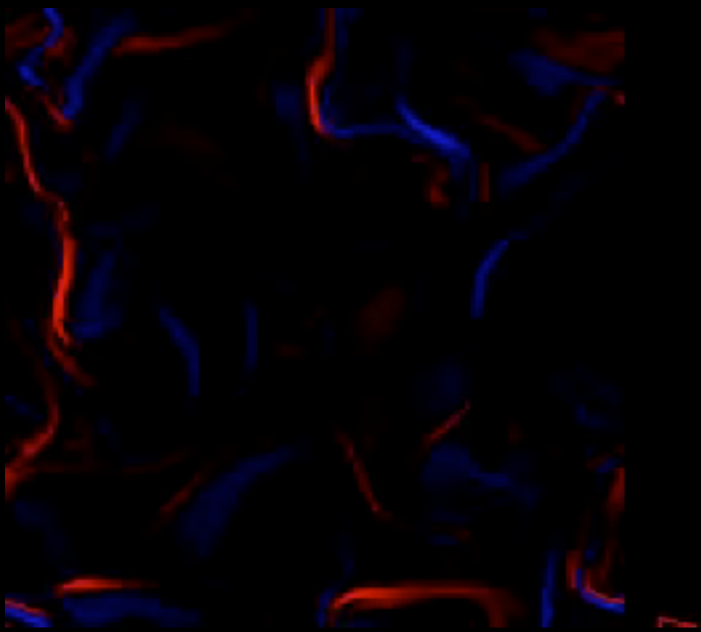


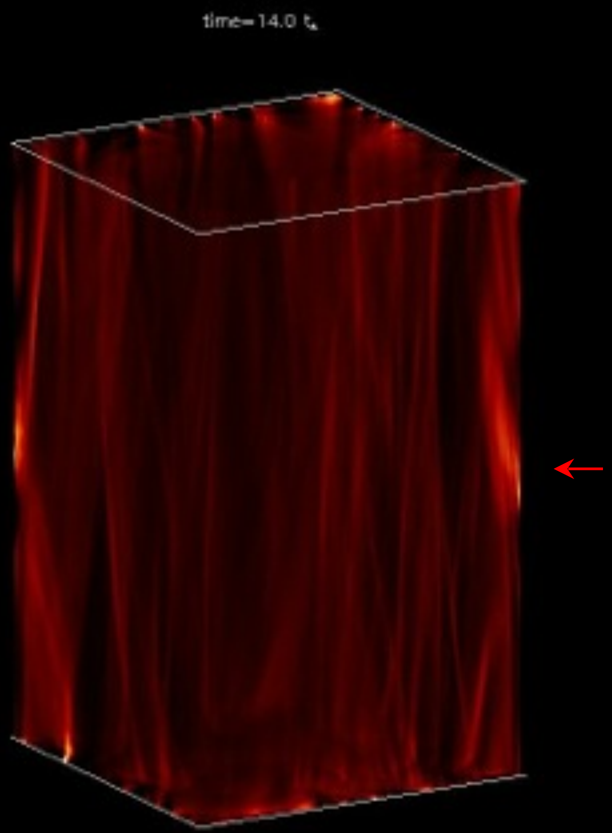
$$E_k \approx \frac{u_k^2}{k} = \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$

Kolmogorov spectrum (K41)

Current sheet formation

- Most of the energy dissipation takes place in current sheets. We display the current density (upflows & downflows) along the loop in a transverse cut.

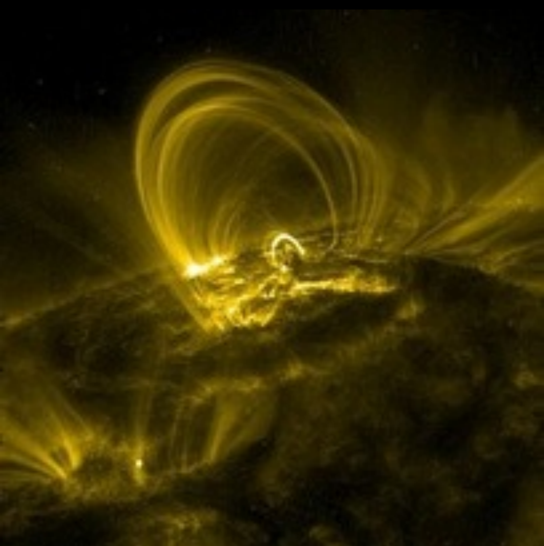


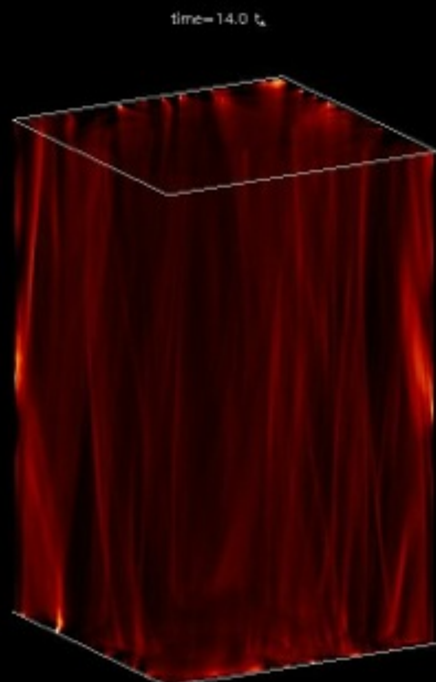


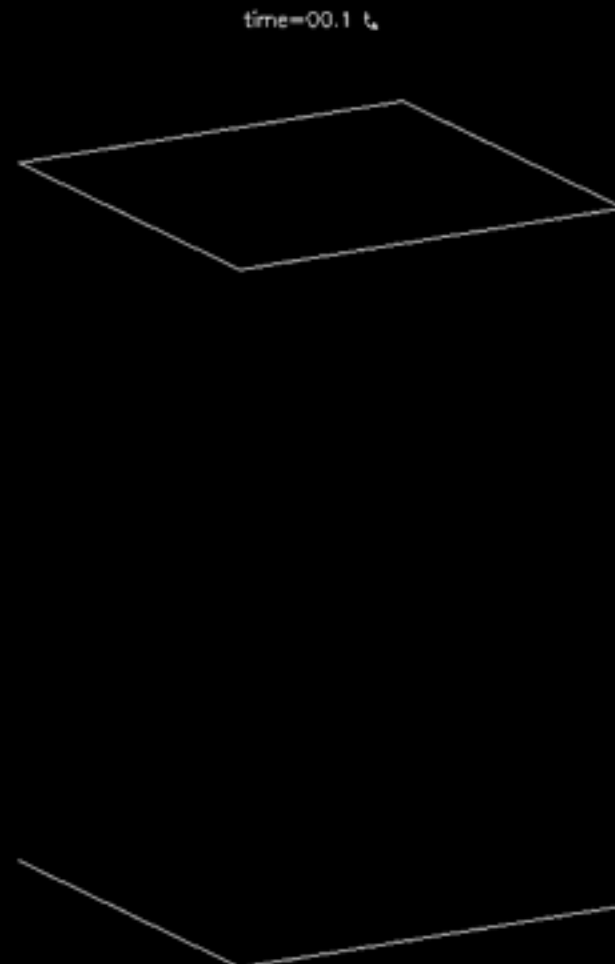
- Versus height.
- Versus time.

Current sheets in 3D

- 3D distribution of the energy dissipation rate.
- We display the dissipation rate during 20 Alfven times with a cadence of 0.1 t_A.

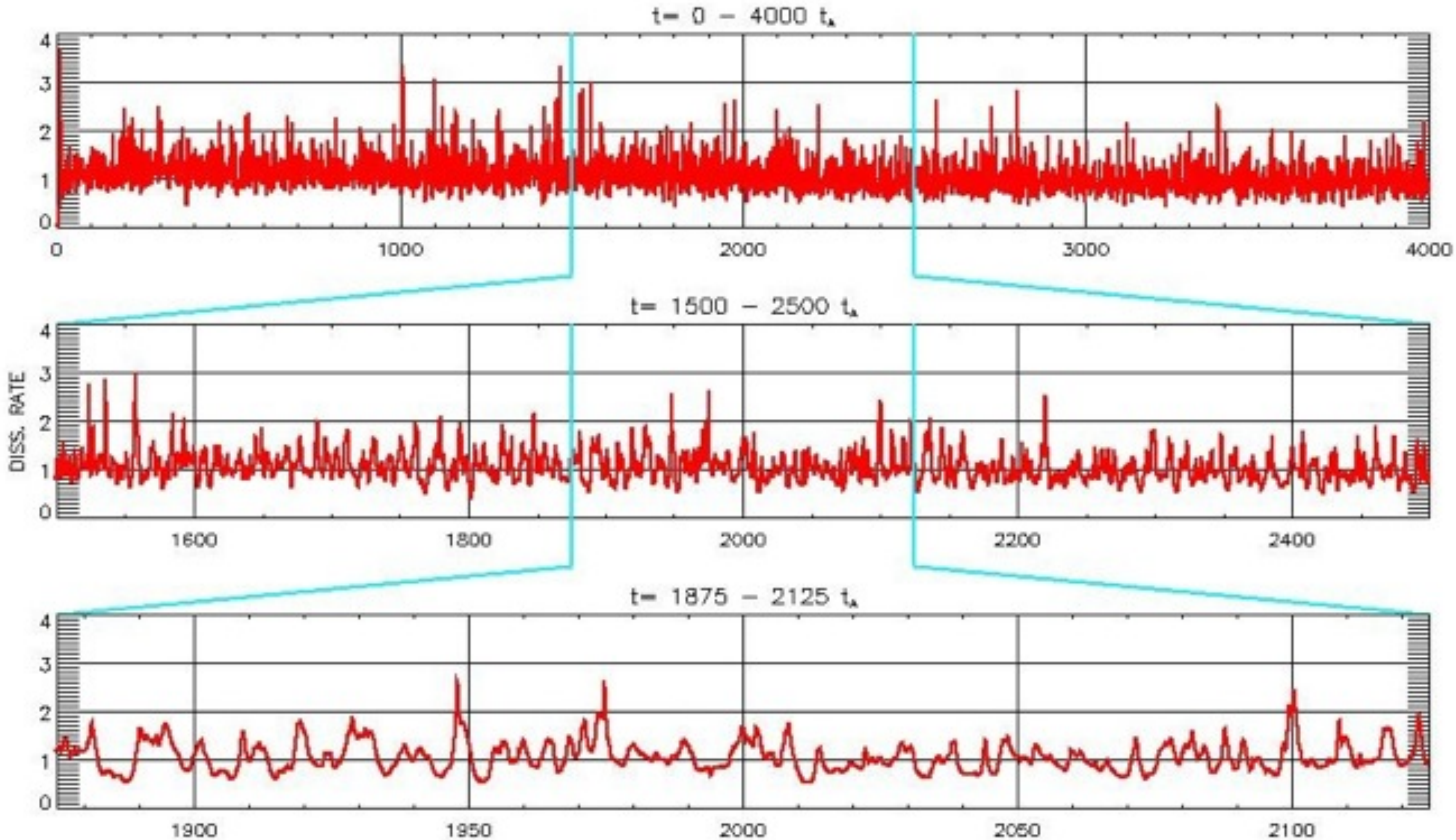






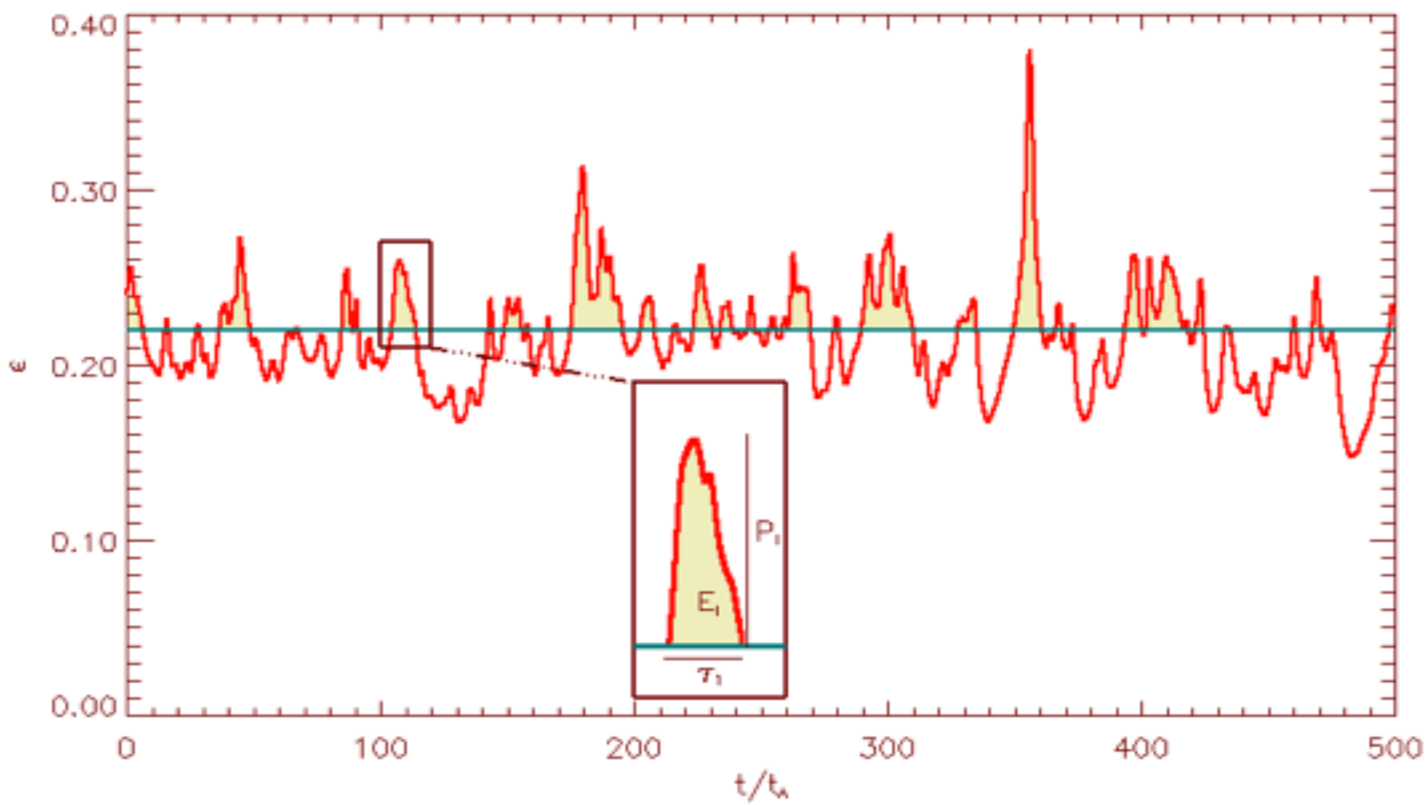
Energy dissipation rates

- The complete time series of energy dissipation rate is displayed below. It shows a mean value (consistent with the scaling law given above) plus a rather spiky structure.



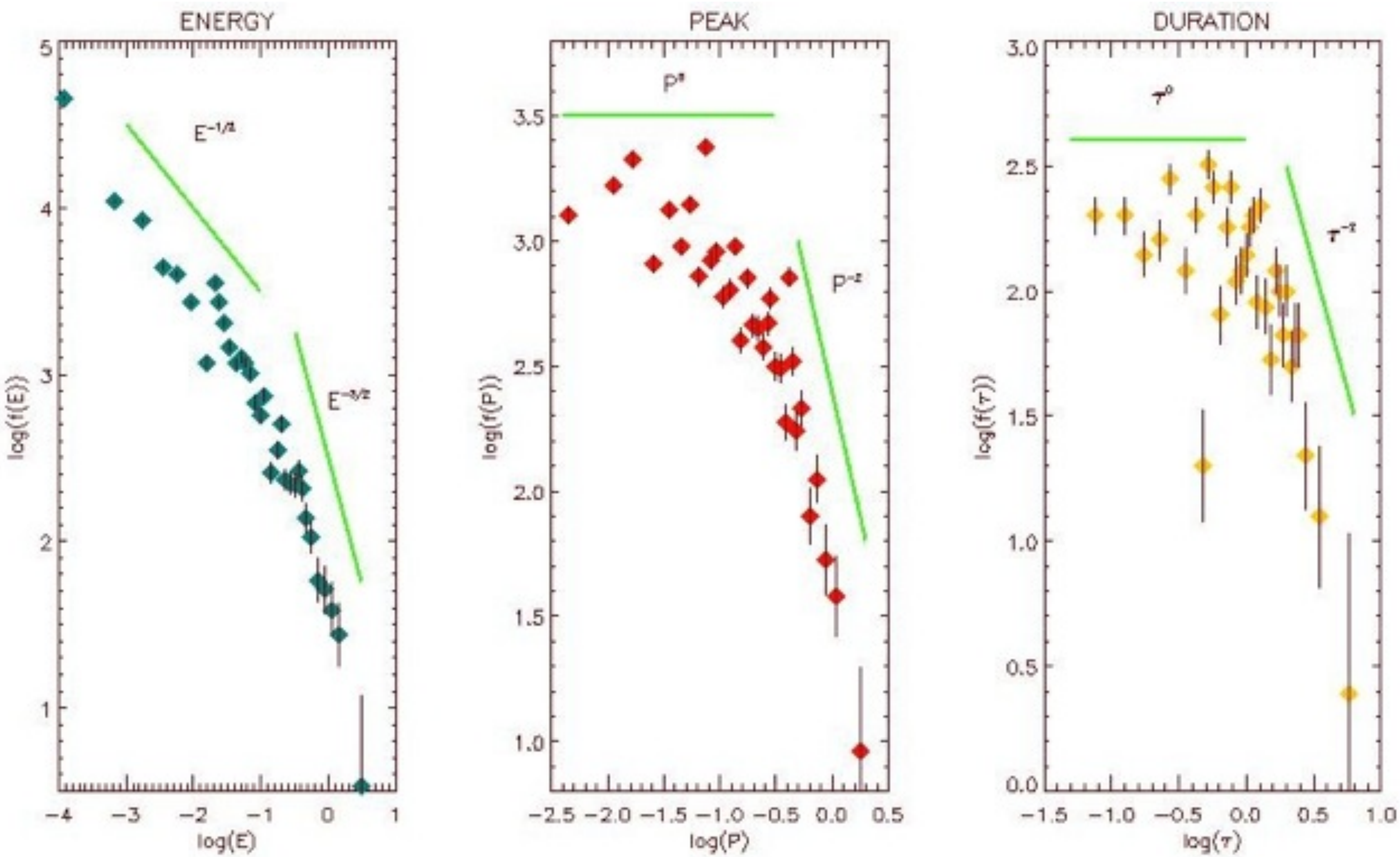
Dissipative events

- We associate the intermittent bursts of dissipation with Parker´s nanoflares (Parker 1988).
- To carry out our statistical analysis, we draw a threshold and count as events everything that pops up above that line.
- For each event, we compute its peak dissipation (P_i), its total dissipated energy (E_i), and its duration (T_i).
- We compute histograms for these quantities and look for correlations between them.



Histograms of dissipation events

- Histograms of energies, peak dissipation rates and durations display a power law behavior. We estimated the slopes using standard fitting techniques.



Observed power laws

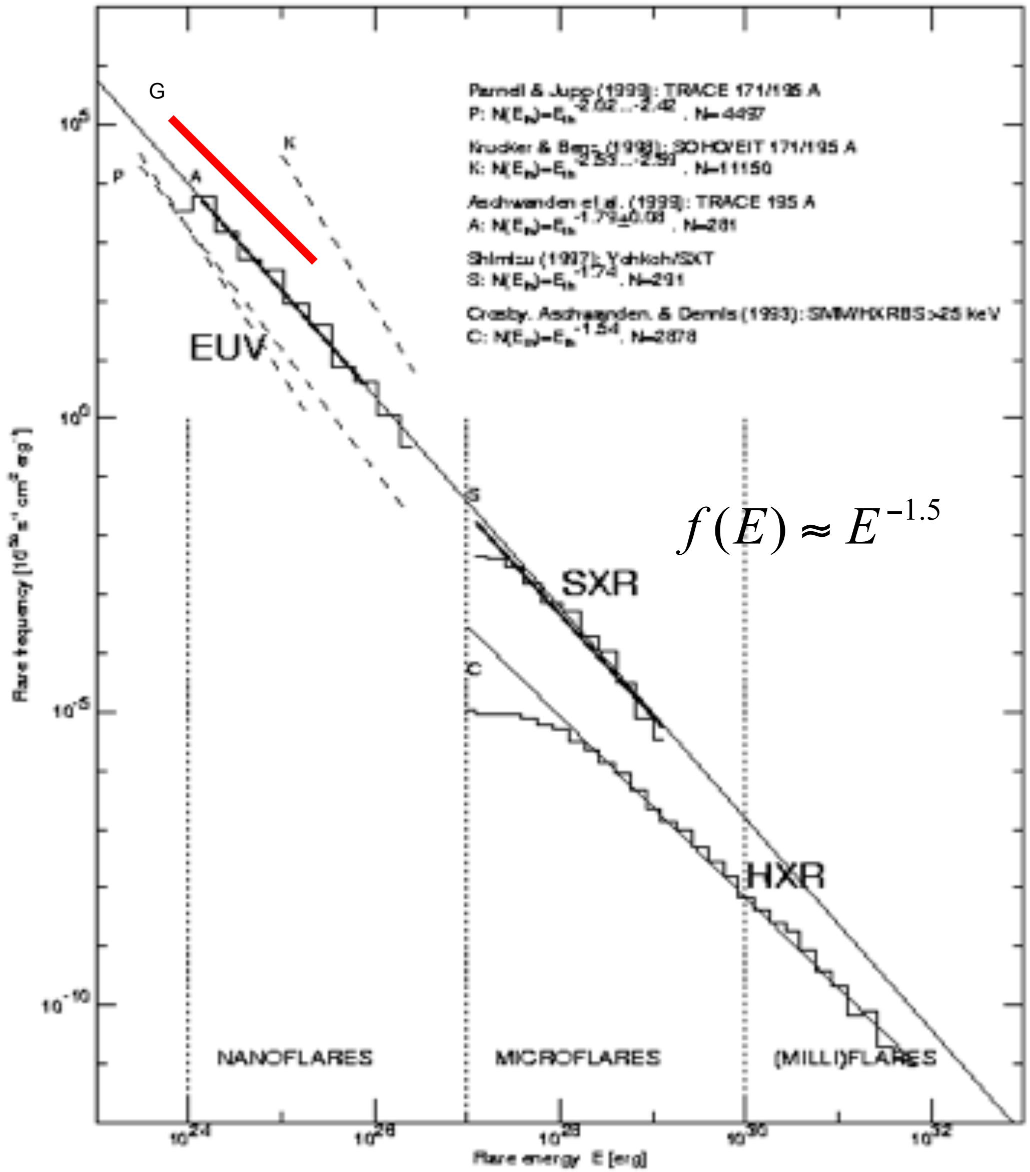
- It is also consistent with the energy distributions derived from observations (Aschwanden 2000).

- The total energy dissipation rate can be simply obtained as

$$\varepsilon = \int_{E_{\min}}^{E_{\max}} dE\, E\, f(E)$$

- For slopes flatter than –2 (which seems to be the case) the dissipation is dominated by the large events.

- For slopes steeper than –2, small events dominate instead.



Conclusions

- We introduced the MHD equations, which is an adequate theoretical framework to describe the large scale behavior of a number of astrophysical and laboratory applications.

- We also presented to so called reduced approximation, which is appropriate for plasmas embedded in relatively strong magnetic fields.

- As an astrophysical application, we showed RMHD simulations to study the internal dynamics of magnetic loops in the solar corona.

- The development of a turbulent MHD regime in coronal loops increases the heating rate to levels comparable to the radiative cooling of these loops.

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Simulations: spatial integration

➢ We focus on Fourier-Galerkin methods. Let us illustrate on Burgers equation

$$\partial_t u + u \partial_x u = \nu \, \partial_{xx} u$$

for u(x,t) on the interval $0 \le x < 2\pi$ assuming periodic boundary conditions and the initial condition $u(x,0) = u_0(x)$

➢ We expand in a truncated Fourier expansion ⟹ $u^N(x,t) = \sum_{k=-N/2}^{N/2} u_k(t)\, e^{ikx}$

➢ Demanding zero projection of the solution u(x,t) on the truncated Fourier space

$$\partial_t u_k = -(u\partial_x u)_k - \nu\, k^2\, u_k \quad , \quad (u\partial_x u)_k = \sum_{l+m=k} im u_l u_m$$

➢ This truncated expansion $u^N(x,t)$ converges exponentially fast to the exact solution as $N \to \infty$

However, it is computationally very demanding, it involves $O(N^2)$ operations.

Simulations: spatial integration

➢ The FFT algorithm yields the discrete set $\{\hat{u}_k\}$ from the set $\{u(x_j)\}$ after $O(N \log N)$ floating point operations.

$$\left\{u(x_j)\ ,\ x_j = \frac{2\pi}{N} j\ ,\ j = 0,\dots,N-1\right\} \quad \overrightarrow{FFT} \quad \left\{\hat{u}_k\ ,\ k = -N/2+1,\dots,N/2\right\}$$

➢ The strategy of computing spatial derivatives in Fourier space and nonlinear terms in physical space, is known as pseudo-spectral, i.e.

$$\partial_t u_k = -(u\partial_x u)_k - \nu\, k^2\, u_k \quad , \quad (u\partial_x u)_k = FFT(FFT^{-1}(u_k)\ FFT^{-1}(iku_k))$$

➢ The relation between discrete Fourier coefficients $\{\hat{u}_k\}$ and the continuous ones is

$$\hat{u}_k = u_k + \sum_{m\neq 0} u_{k+Nm}$$

➢ This sum causes a spurious effect known as aliasing when computing nonlinear terms. Aliasing effects can be suppressed by applying the "two-thirds rule", i.e.

$$\hat{u}_k = 0 \quad , \quad \forall \quad |k| \geq N/3$$

Simulations: temporal integration

➢ We advance the solution through discrete time steps ⟹ $t_i = i\Delta t$

➢ In compact notation, if $\frac{dU}{dt} = F(U,t)$

where F is a nonlinear and spatial differential operator, we use a second order Runge-Kutta scheme.

➢ We first advance half a step ⟹ $U^{i+\frac{1}{2}} = U^i + \frac{\Delta t}{2} F(U^i, t_i)$

and use $U^{i+\frac{1}{2}}$ to jump the whole step ⟹ $U^{i+1} = U^i + \Delta t\, F(U^{i+\frac{1}{2}}, t_{i+\frac{1}{2}})$

➢ This is second order accurate (i.e. $O((\Delta t)^2)$). The size of the step is limited by

the CFL condition, i.e $\Delta t \leq \Delta x / u_0$ for $\partial_t u = u_0 \partial_x u$